Precision beats power, timing beats speed.
Conor McGregor

It's time

Inside a uniform globe of radius R, the tension of gravitational field on the surface of which is equal to g, a smooth and narrow channel is cut connecting the north pole of the sphere with the point of its equator. The channel is built in such a way that if a point particle is released from the North Pole without an initial velocity, then the time of its movement along the channel to the equator point turns out to be the least possible. Within the framework of this task, you have to find this minimum possible time.

1. *(1 point)* Find the speed of a point particle v at a distance r from the center of the globe.

Further, it is most convenient to use Hamilton's optico-mechanical analogy. Let a substance with a refractive index $n(r)$ be distributed spherically symmetrically inside the globe. Consider the motion of a ray in such an environment. Let us denote the angle between the radius vector of the ray $\vec{r}$ drawn from the center of the globe and the vector of its velocity $\vec{v}$ by φ.

2. *(2 points)* Find the relationship between n, r and φ.
3. *(1 point)* Find, up to a constant factor, the refractive index $n(r)$, such that the optimal trajectory of a point particle and the trajectory of a ray in the given medium coincide.

Let O denote the center of the globe, and C — a point of the trajectory that the ray is currently passing. Let us draw through the point C a chord AB (in the plane of the ray trajectory) being perpendicular to the direction of the ray velocity vector $\vec{v}$, with $AC > BC$.

4. *(3 points)* For a fixed position of point B, find a locus (all possible geometrical positions) of points C. Also find the minimum possible distance OC.
   Note. You may need an intersecting chords theorem.

Let's go back to the original problem.

5. *(3 points)* Find the minimum possible time $t_{\min}$ for a point particle to travel from the North Pole to a point on the equator.

Problem's author: Andrey Uymin.

First hint — 17.05.2021 14:00 (GMT+3)
Second hint — 19.05.2021 14:00 (GMT+3)

End of the third tour — 21.05.2021 22:00 (GMT+3)