LPR VI Cup

10.s06.e02

– That's impossible!
– That's science!
Back to the Future

Time

Time passed.

And so, the usually not too crowded workshop of Hans felt especially empty this rainy morning. Perhaps the reason was the uncharacteristic disorder for this place? Crumpled sheets with calculations and diagrams understandable only to their author were scattered all over the floor, and once carefully kept tools and instruments were chaotically strewn without any system on the tables and shelves. Even the books, usually neatly arranged on the shelves, were now scattered haphazardly, and some were even lying open, with dog-eared yellowed pages and coffee cup stains on them.

Anyone who knew Hans, after one look at the state of the workshop, would first want to report a robbery. But, as in the openings of classic detective stories, nothing had disappeared from the workshop~~, except for Hans himself~~. A sudden gust of wind slammed the window shut, and the sound of rustling scattered drafts, like the noise of the rain outside, vanished into the street.

Only the ticking of the clock remained in the workshop.

Introduction

In this task you will study the structure of a wristwatch.

The mechanism of its operation can be divided into five components, which are shown in Figure 1.

Due to energy dissipation, any mechanical system requires an energy source. For example, in wall clocks (pendulum clocks with cuckoo), the energy comes from a weight-driven mechanism. In wristwatches, however, the energy source is a spring motor. When the watch is wound, motion is transmitted to the arbor (1), which is attached to one end of a flat spring (2), shaped like a spiral. The spring twists and stores energy. The other end of the spring is fixed to the wall of the barrel drum (3), which then transfers the stored energy to the gear train (wheel system) connected to the minute, second, hour ~~, and year~~ hands (not shown in the figure).

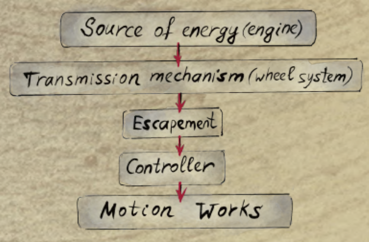

Figure 1: Diagram of a clock mechanism

To regulate the rotation of the gears (and thus the movement of the hands), a trigger and a regulator are used, preventing the wheel system from spinning uncontrollably. They consist of three parts:

- Escapement, or a pallet fork, is a lever.
- A trigger wheel is a toothed wheel that pushes the escapement.
- The controller is a spring that compresses and decompresses at a certain frequency.

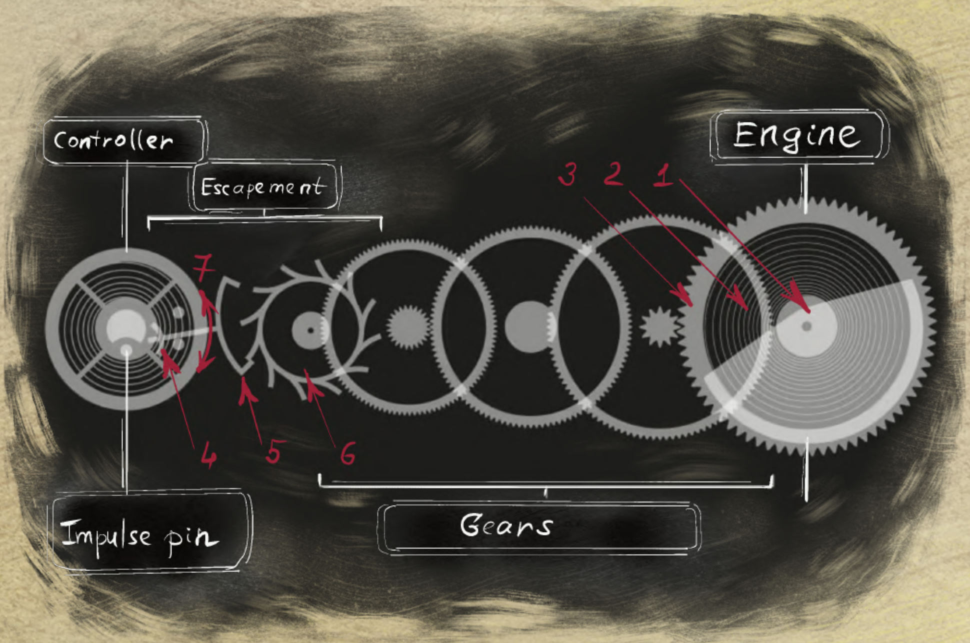

Figure 2: Diagram of a clock mechanism. 1 – shaft, 2 – mainspring, 3 – barrel drum, 4 – fork horn, 5 – pallet, 6 – escape wheel, 7 – direction of fork movement. The mechanical movement takes place in two parallel planes, which are themselves parallel to the plane of the drawing. The controller is located in the first plane, while the escapement, gears, and engine are located in the second plane. Interaction between the planes occurs only through the impulse pin.

The pallets (5), mounted on the escapement, hold the trigger wheel (6) in place until the escapement shifts, allowing the wheel to advance by one tooth. The movement of the escapement is possible because the impulse jewel, mounted on the controller, periodically strikes the fork horns (4), alternating between the left and right sides.

Part I. A spiral spring

The spiral spring is a central component of a wristwatch, as it is used both in the controller and the spring motor, though their functions differ. In the controller, it acts as a harmonic oscillator, while in the motor, it serves solely as an energy storage device, compensating for losses in the mechanical system.

In this section, you will determine the torsion modulus of such a spring and the oscillation period of the controller.

Consider a spiral spring resting on a smooth horizontal table. It is made of a steel strip with thickness b, width h (along the axis perpendicular to the table), lenght L, and Young's modulus E.

If a torque is applied to both ends of this spiral, it will twist (or untwist) by an additional angle relative to its equilibrium position. The torsion modulus of the spring is defined as the proportionality coefficient between the applied torque and this additional twist angle.

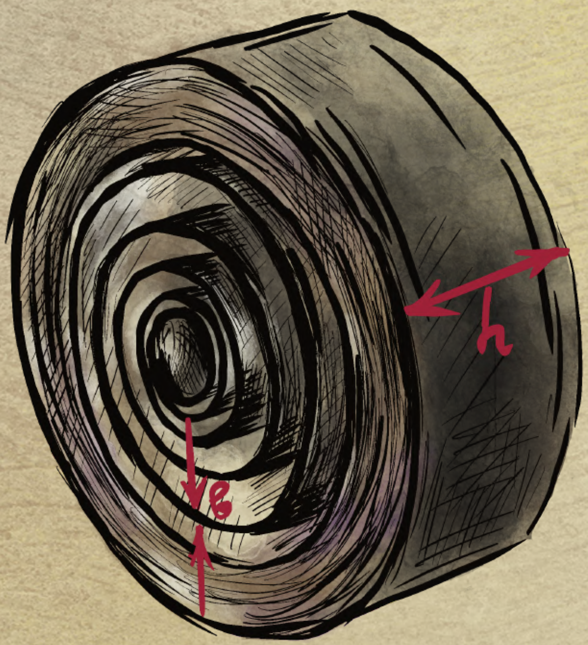

Figure 3: Spring in the free state

1. *(1 point)* Assuming it is known that the torque twisting the spring is directly proportional to the angle of rotation (relative to equilibrium) to the first power, and using the extended dimensional analysis method (see here), find the torsion modulus of such a spring.
2. *(1 point)* Mass is a measure of inertia in translational motion. If a body rotates, then instead of mass, the concept of the moment of inertia I of the body is used. When a spiral spring rotates around an axis passing through its center and perpendicular to the plane of its coils, its moment of inertia is given by $I = m\rho^2$, where m is the mass of the spring, and ρ is some effective radius of inertia of the spring. Using extended dimensional analysis, find the formula for the period of rotational oscillations of such a spring, up to a constant factor.

Part II. The Balance Wheel

In an ideal mechanical system, the controller performs periodic motion with a constant period and without damping, ensuring the steady running of the clock.

In reality, this is not the case, and there are many factors that cause a clock to run slow or fast. For example, thermal expansion or impact. In this section, you are invited to study these effects. ~~Our job is to make an offer. Your job is to refuse.~~

Thermal Expansion

Assuming that as a result of thermal expansion all lengths of the spring have increased by 2%, find:

1. *(0,5 points)* By what percentage will the period of oscillation of such a clock change;
2. *(0,5 points)* The absolute error accumulated by such a clock over one day of real time.

Impact

To analyze the effect of an impact, we will consider not a spiral spring but a simpler oscillator model — a mass on a spring, — whose motion is entirely analogous to the controller in a wristwatch. However, this model is significantly simpler and more familiar for analysis.

Consider a mass m on a smooth horizontal surface, oscillating on a horizontal spring with stiffness k. Let x and v denote the displacement from equilibrium and the velocity of the mass, respectively, and let A represent the maximum displacement (amplitude of oscillations). In this case, the position of the mass as a function of time is given by:

$$x(t) = A\cos(\omega t + \varphi_0),$$

where $\omega = \sqrt{k/m}$ is the angular frequency of harmonic oscillations, the expression in parentheses is called the phase of the oscillations, and φ_0 is the initial phase.

1. *(0 points)* ~~Thanks for not giving negative points~~ Find the velocity of the mass as a function of time.

Suppose the mass is performing such oscillations when, at some moment, it is subjected to a brief impulse (i.e., an impact whose duration is much shorter than the oscillation period). Assume the magnitude of the velocity change is $\Delta v < v_{max}$, where v_{max} is the maximum speed of the mass before the impact, and the impulse is applied along the line of oscillation. Determine how this impact alters the period of motion in the following cases:

1. *(0,5 points)* The impact occurs at the moment of maximum displacement from equilibrium. Assume the mass is pushed in the same direction as its motion just before the impact. How does the answer change if the velocity is imparted in the opposite direction? Express your answer in terms of Δv, v_{max}, ω.
2. *(0,5 points)* The impact occurs when the mass*passes through the equilibrium position. Assume the mass is pushed in the same direction in the same direction as its motion at the moment of impact. Express your answer in terms of Δv, v_{max}, ω.
3. *(0,5 points)* The impact occurs at a moment when the velocity divided by the oscillation frequency is $\sqrt{3}$ times greater than the displacement. Assume the mass is pushed in the same direction as its motion at the moment of impact. Express your answer in terms of Δv, v_{max}, ω.

Aside from timing inaccuracies, mechanical watches may stop due to energy dissipation caused by friction in the system. We will also analyze energy loss processes using the mass-spring model.

Dry Friction

Consider a mass $m = 100$ g on a horizontal surface, oscillating on a horizontal spring with stiffness $k = 100$ N/m. The coefficient of friction between the mass and the surface is $\mu = 0.4$. Taking $g = 10$ m/s^2, find:

1. *(0,3 points)* The size of the region, in the case where the mass stops, within which it will not resume its motion.

2. *(0,3 points)* Suppose the mass is displaced by a distance $A = 25.5$ cm and released without initial velocity. How many complete oscillations will the mass make before coming to a complete stop?

3. *(0,3 points)* What distance will the mass travel from the start of motion until it comes to a complete stop?

Viscous Friction

Let the same mass be on a smooth horizontal surface and oscillate on the same spring. However, it is now affected by a viscous friction force such that it is proportional to the speed of the load. In this case, the law of motion is described by the equation:

$$x(t) = Ae^{-\gamma t} \cos \Omega t,$$

where A is the amplitude of oscillations, γ is the damping coefficient, and $\Omega = \sqrt{\omega^2 - \gamma^2}$ is the frequency of such oscillations. Assuming $A = 25.5$ cm, $\gamma = 0.01$ s^{-1}, find:

1. *(0,3 points)* The ratio of the energy stored in the oscillator to the losses over the period.

2. *(0,3 points)* The distance that the body will take in the first 500 of full oscillations.

Part III. Barrel Drum

In the engine, a spiral spring is wound in the shape of a snail so that one end is attached to a shaft and the other to the barrel drum. In the figure, the spring is shown fully unwound on the left and fully wound on the right. In the first case, the number of turns is n_1, and in the second case, the number of turns is n_2. For a large number of turns of the spiral, the torque it produces can be considered proportional to this number of turns. Therefore, in order to store as much energy as possible in the fully wound spring—and thus reduce how often the clock needs to be wound—the aim is to maximize the number of revolutions.

In this section, you are offered ~~and you can still refuse~~ to find out which geometry implements this case.

Consider ρ_2 is the outer radius of the fully wound spring, ρ_1 is the inner radius of the unwound spring, r is the arbor radius, R is the barrel drum radius, b is the spring thickness, L is the length.

1. *(1 point)* Given R, r, and b as known, determine the ratio ρ_1/ρ_2 that maximizes the number of barrel rotations until complete unwinding.

2. *(0,25 points)* Derive the dependence $R(b,\ L,\ r)$ for this case.

3. *(0,25 points)* Using the previous results, determine the maximum number of barrel rotations n_{max}, assuming that in the real mechanism $r = R/3$ and the spring thickness $b = R/45$.

In real clocks, the dependence of torque on the number of barrel revolutions is not linear. Figure 6 shows a graph of this experimental relationship. The upper curve represents the winding of the spring, while the lower one represents its unwinding.

9. *(points)* Using the experimental dependence, determine the efficiency of the spring. η.

Figure 4: Position of the spring in the barrel in the unwound state

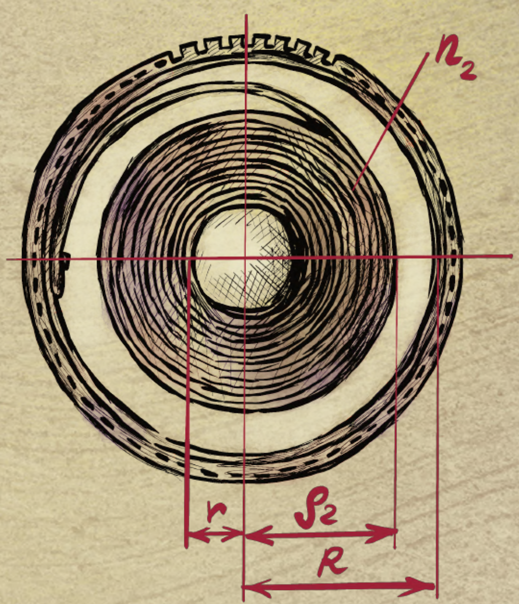

Figure 5: Position of the spring in the barrel in the wound state

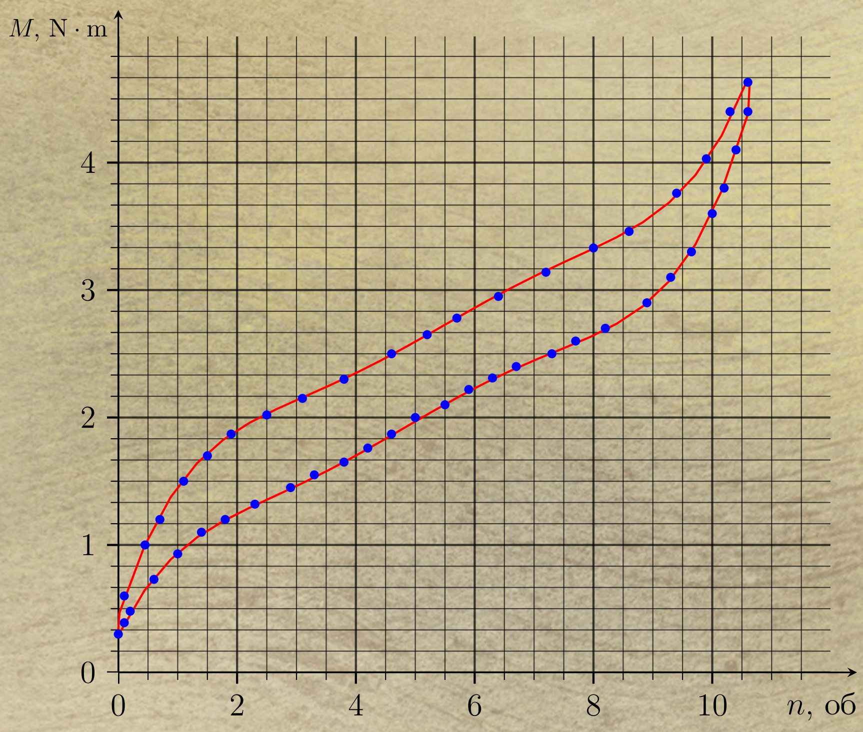

Figure 6: Experimental dependence $M(n)$ graph

Part IV. Escapement

In this part, you are still asked to consider the mechanics of the watch's escapement process.

We will analyze the motion in two planes: the plane of the controller and the plane of the anchor, wheel system and engine. These planes are interconnected through the impulse pin, which crosses both planes and is rigidly fixed to the controller's disk at a certain distance from its axis of rotation (see Fig. 2).

The controller generates a periodic signal, the frequency of which will later be used to set the time scale interval.

The escapement, gears, and engine are used so that, at the moment the controller passes through its equilibrium position, the escapement briefly comes into contact with it via the impulse pin, thereby "replenishing" its energy without significantly altering the period.

In more detail, the operating cycle of the mechanism proceeds as follows:

Unlocking phase:

- The impulse pin moves counterclockwise and enters the fork horn.
- The first contact occurs between the impulse pin and the inner surface of the fork horn.
- The rotation of the escapement releases the escape wheel.

Impulse phase:

- The released gear wheel begins to rotate under the torque of the wound spring.
- Almost simultaneously, the second and third collisions occur in the escapement system, accompanying the start of the controller's acceleration. The pallet fork transmits torque to the controller via the impulse pin, resulting in the impulsive acceleration of the oscillating part.

Drop phase:

- The impulse pin exits the pallet fork.
- Almost simultaneously, the fourth and fifth collisions occur, accompanying the locking of the gear wheel by the pallet through contact with its next tooth, and the braking of the pallet against the limiting stop.

Further, the oscillating mechanism, released and receiving a portion of energy from the engine, continues its harmonic movement, reaching the position of maximum deviation, after which it unfolds, and then the same acceleration process repeats with the same impacts, moving in the opposite direction, with precision until symmetrically reflected escapement sections are used in the process of impacts.

1. *(1 point)* Assuming that the torque of the driving force in the mainspring is constant and equal to M, the effective torque transmission coefficient to the controller through the gear wheel, gears, escapement, and impulse pin between impacts 3 and 4 is n, the total moment of inertia of the balance wheel is I and is much greater than the moments of inertia of the gear wheel, gears, and escapement, the torsion modulus of the controller spring is k, the angular amplitude of oscillations is φ, and the time between the third and fourth impacts is t, determine what fraction of the energy stored in the oscillatory motion is lost by the system in one period of oscillation.
2. *(0,5 points)* Using the attached audio file, determine to an accuracy of 0.001 s the minimum time interval between all phases of the mechanism's working cycle. Describe and justify your reasoning process. For audio file analysis, we recommend using Audacity or similar software.

First Hint — 05.05.2025 20:00 (Moscow time)
Second Hint — 07.05.2025 12:00 (Moscow time)

Final of the first round — 09.05.2025 20:00 (Moscow time)