



*There's a difference between knowing the path and walking the path.*

*The Matrix (1999)*

## Persephone

### Introduction

In the Fifth Episode of the Second Season of the LPR Cup, you will be asked to explore diverse optical systems using  $2 \times 2$  matrices. A video explaining how to work with such matrices is available at [link](#). If you have any questions on how to work with matrices correctly, you can ask [Abay](#).

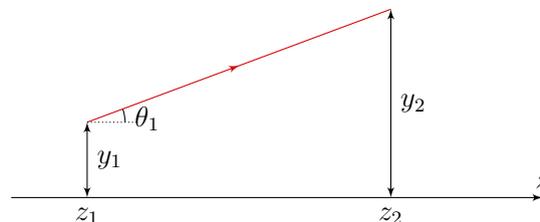
**Important!** You can ask questions only on how to work with matrices. Questions related to the task should be asked in the private messages [of the LPR Cup](#).

### General theory

Let us call an optical system *centered* if the centers of curvature of all spherical refractive and reflecting surfaces are located on a single straight line, which is called *the main optical axis*. If all the beams propagating in the system are at small distances from the optical axis and form small angles with the axis, let us say that the *paraxial approximation* is accurate.

**Note.** In this problem, unless otherwise specified, let us assume that the paraxial approximation is accurate, and all optical systems are centered.

Let us introduce a Cartesian coordinate system: the  $Oz$  axis, which coincides with the main optical axis;  $Ox$  and  $Oy$  axes, which are perpendicular to the main optical axis, with  $Oy$  axis lying in the drawing plane. Consider a beam of rays propagating in the drawing plane. At any point with a known coordinate  $z$ , a ray can be uniquely determined if its distance to the optical axis and the angle  $\theta$  that this ray forms with this axis are known. For example, the figure shows a ray that passes through a point at a distance  $y_1$  from the optical axis and forms an angle  $\theta_1$  with this axis (see fig.). Let us measure the angle  $\theta$  in radians and consider it *positive* if it *corresponds to a counterclockwise rotation* from the positive direction of the  $z$  axis to the direction in which the light propagates along the ray.



Although the distance  $y$  and the angle  $\theta$  are obvious parameters for setting the position and direction of the ray propagation, two other parameters are more often used in the literature: the distance  $y$  and *optical directional cosine*  $v = n \cdot \theta$ , where  $n$  is the refractive index of the

medium at a given point. In the future, let us characterize the ray with this pair of numbers and say that it is unambiguously characterized by the following vector

$$\begin{pmatrix} y \\ n\theta \end{pmatrix} \equiv \begin{pmatrix} y \\ v \end{pmatrix}.$$

When light propagates in an optical system, three processes can occur with a beam: *propagation process*, *refraction of light at the interface of two media*, and *reflection of light*. For each process, let us match *ABCD* with a matrix by which we will multiply the vector that defines the ray in the plane  $z = \text{const}$ , as a result, we will get a new vector that corresponds to the new location of the ray. As an example, consider the process of ray propagation in a homogeneous medium.

### Matrix of propagation $T$

The figure above shows the process of ray propagation in a homogeneous medium with a refractive index of  $n$ . Consider two planes with coordinates  $z_1$  and  $z_2$ . It is clear that the angle between the ray and the main optical axis in both planes is the same, so

$$\theta_2 = \theta_1 \iff v_2 = v_1,$$

where  $v_1 = n\theta_1$  and  $v_2 = n\theta_2$ .

On the other hand, the coordinate  $y_2$  can easily be written in terms of  $y_1$  and  $\theta_1$ . Indeed:

$$y_2 = y_1 + \text{tg } \theta_1(z_2 - z_1) \approx y_1 + \theta_1(z_2 - z_1) = y_1 + v_1 \frac{z_2 - z_1}{n}.$$

From the last two equations, we can get that the equation of ray propagation in a homogeneous medium can be written as

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{z_2 - z_1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

And the *ABCD* matrix of propagation is

$$T = \begin{pmatrix} 1 & \frac{z_2 - z_1}{n} \\ 0 & 1 \end{pmatrix}$$

If the ray is involved in several processes in a row, some transformations should be done with it. These transformations are equivalent to matrices multiplication. Indeed, if the ray was at a distance  $y_1$  from the optical axis and propagated at a distance  $l_1$  along it, this is equivalent to multiplying the vector with the components  $y_1$  and  $v_1$  by the corresponding matrix of propagation  $T_1$

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{l_1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

If the ray continued to propagate in a homogeneous medium for an additional distance  $l_2$ , then this is equivalent to multiplying the vector with the components  $y_2$  and  $v_2$  by the matrix  $T_2$

$$\begin{pmatrix} y_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & \frac{l_2}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{l_2}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{l_1}{n} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = T_2 T_1 \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = T \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

Thus, we can say the final transformation matrix  $T$  is equal to the product of two propagation matrices written in the **reverse** order. There we used the fact that the products of matrices are associative. So, the following statement is true:

$$ABC = (AB)C = A(BC)$$

As an exercise, make sure that the matrix  $T$  has the form

$$\begin{pmatrix} 1 & \frac{l_1 + l_2}{n} \\ 0 & 1 \end{pmatrix}.$$

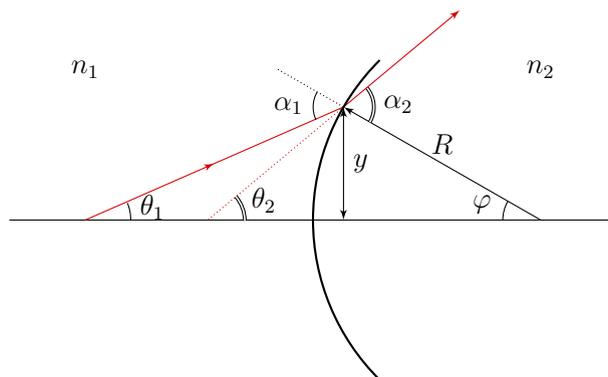
Note that in this case the relation  $T_1 \times T_2 = T_2 \times T_1$  is true. In other words, the matrices commute, which is not always true, including the examples that we will discuss later. Therefore, the order of writing the matrices is very significant! And in our case, the matrices are written in the **reverse order**!

### Matrix of refraction $P$

Consider a spherical interface between two media with refractive indexes  $n_1$  and  $n_2$ . Let the radius of curvature of the surface be positive if the angle between the axis  $Oz$  and the radius-vector which connects the center of curvature and the spherical surface is obtuse. If this angle is acute, then let this radius of curvature be negative (see fig.).



Let us consider the refraction of light on a spherical surface and find the matrix of refraction  $P$ . Let the ray pass from a medium with a refractive index  $n_1$  to a medium with a refractive index  $n_2$  (see fig.).



It is clear that the  $y$  coordinate does not change when the ray crosses the interface between the two media, so

$$y_2 = y_1.$$

Let the angles of incidence and refraction be  $\alpha_1$  and  $\alpha_2$ , respectively, and the angles between the optical axis and the incident and refracted rays –  $\theta_1$  and  $\theta_2$ . The figure shows that  $\alpha_1 = \theta_1 + \varphi$ , and  $\alpha_2 = \theta_2 + \varphi$ , where  $\varphi$  is the angle between the optical axis and the radius to the point where the ray is refracted. Let us write the Snell's law  $n_1\alpha_1 = n_2\alpha_2$  and use the fact that  $\varphi = y/R$ , then

$$n_1 \left( \theta_1 + \frac{y}{R} \right) = n_2 \left( \theta_2 + \frac{y}{R} \right).$$

Rewriting the last equation in terms of the directional cosine  $v_1$  and  $v_2$ , we get that

$$v_2 = \frac{n_1 - n_2}{R} y_1 + v_1,$$

and then we find

$$\begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{n_2 - n_1}{R} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ v_1 \end{pmatrix}.$$

In the lower-left corner of the matrix, let us take out a sign and select a fraction, which is called the optical power of the surface  $P_1$

$$P_1 = \frac{n_2 - n_1}{R}.$$

Thus, we get that the matrix of refraction has the form

$$P = \begin{pmatrix} 1 & 0 \\ -P_1 & 1 \end{pmatrix}.$$

**Exercise 1.** Show that a thin biconvex lens with radii of curvature  $R_1 > 0$  and  $R_2 < 0$  and a refractive index  $n$ , placed in a medium with a refractive index  $n_0$ , has the following matrix of transformation of optical rays

$$\begin{pmatrix} 1 & 0 \\ -(P_1 + P_2) & 1 \end{pmatrix},$$

where  $P_1 + P_2 = (n - n_0) \left( \frac{1}{R_1} - \frac{1}{R_2} \right) = \frac{1}{F}$ .

**Exercise 2.** Find the optical power of a thin biconvex lens with radii of curvature  $R_1 > 0$  and  $R_2 < 0$  and a refractive index  $n$ , if it is placed between two media with refractive indexes  $n_1$  and  $n_2$ .

**Exercise 3.** Prove that the optical powers of two lenses which are close to each other add up.

# Problem

## Part 1

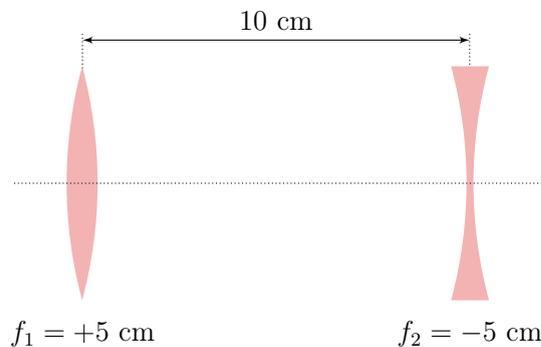
Let there be some optical system, which is described by some  $ABCD$ -matrix that transforms a ray outgoing from the plane with coordinate  $z_1$  into a ray entering the plane  $z_2$ . The parameters of the optical system were selected so that one of the matrix elements became equal to zero. What physical property does the system have if

1.  $A = 0$ ;
2.  $B = 0$ ;
3.  $C = 0$ ;
4.  $D = 0$ .

**Note.** Each of the points weighs zero points, but you can send them, and they will be checked in the CPI format so you can make the right conclusions from your reasoning.

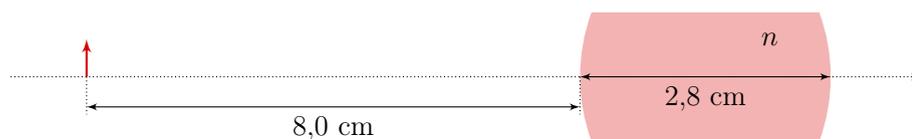
## Part 2

5. (0,5 points) It is known that the ray leaves the scattering lens at a distance of 0,5 cm from the main optical axis at an angle of 0,1 rad to the axis. At what angle and distance from the main optical axis the ray falls on the converging lens?



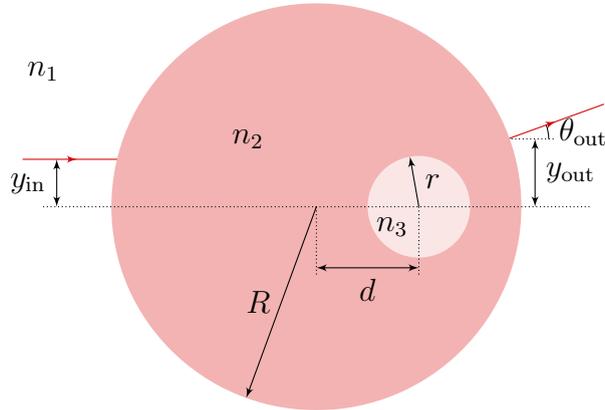
## Part 3

6. (1 point) Both ends of a glass cylindrical rod 2,8 cm long have a spherical shape with a radius of 2,4 cm. Refractive index of the glass is 1,6. An object in the form of a straight line 0,5 cm long is placed on the axis of the rod in a vacuum at a distance of 8,0 cm from the left end of the rod. Find the position and size of the image.



## Part 4

Let's consider the problem from the practice round of the All-Russian Olympiad in physics for school students in 2021. There it was offered to investigate a cylinder of radius  $R$  and with refracting index  $n_2$ . In that cylinder, there a cylindrical cavity of radius  $r$  was made, filled with unknown material with a refractive index  $n_3$  (see fig.). The axes of the cylinders are parallel to each other and placed at a distance  $d$  from each other. The heights of the cylinders are the same. Consider the system parameters as follows:  $R = 7,50$  cm,  $d = 3,50$  cm,  $r = 3,00$  cm,  $n_1 = 1,00$ ,  $n_2 = 1,50$ .



In the problem, it was offered to determine what is greater than  $n_3$  or  $n_2$ . One of the most popular solution options was as follows. The laser beam hits the cylinder parallel to the plane of the cylinders' bases, and parallel to the optical axis passing through the midpoints of the cylinder's axes. Consider that the distance  $y_{in}$  is small enough for the problem to use paraxial approximation. The  $ABCD$ -matrix of this system is found from the equation

$$\begin{pmatrix} y_{out} \\ \theta_{out} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} y_{in} \\ 0 \end{pmatrix}.$$

Here  $A$  and  $D$  are dimensionless quantities,  $B$  is in cm,  $C$  is in  $\text{cm}^{-1}$ .

7. (0,5 points) Find the  $ABCD$  matrix for the given system (numerically).  
**Instruction.** We recommend using WolframAlpha for matrix multiplication.
8. (1,5 points) Plot  $\theta_{out}(n_3)$  graph.
9. (0,5 points) What values of  $n_3$  lead to  $\theta_{out} = 0$ ?
10. (0 points) Is the statement true: «If  $\theta_{out} > 0$ , then  $n_3 < n_2$ »?

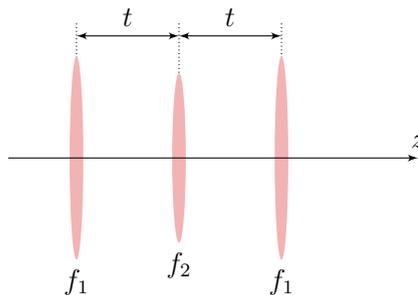
## Part 5

It is known that for certain parameters of the lens system, objects located on the periphery of the space between the lenses become invisible, and the images of objects outside the optical system are not distorted, as if there were no optical system (see fig.).

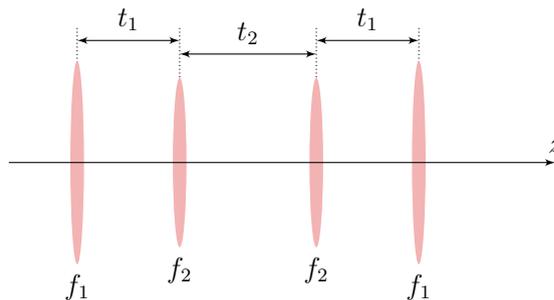


**Note.** In all items of this problem there is **no need to prove** the existence of the invisibility area.

11. (2 points) Show that symmetrical system of three thin lenses with focal lengths distances  $f_1$ ,  $f_2$ , and  $f_1$ , respectively (see fig. below) satisfies the above-described condition only if  $f_1 \gg t$ , where  $t$  is a distance between lenses.



12. (4 points) Find the ratio between  $f_1$  and  $f_2$  focal lengths for a system of four thin lenses with focal lengths  $f_1$ ,  $f_2$ ,  $f_2$ , and  $f_1$  respectively (see fig. below), at which this phenomenon will be observed. Determine at what ratio  $f_1/f_2$  the length of the optical system reaches the extremum. What is the ratio  $t_2/f_2$ ? Consider the distance between the first and second lenses being equal to the distance between the third and fourth lenses.



**Note.** In all the tasks, the distance between lenses and their focal lengths are unknown. The chromatic aberration can be neglected.

First hint — 31.05.2021 14:00 (GMT-2)

Second hint — 02.06.2021 14:00 (GMT-2)

End of the fifth tour — 04.06.2021 22:00 (GMT+3)